RESTRICTED LAZARSFELD-MUKAI BUNDLES AND CANONICAL CURVES

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Dedicated to Professor Shigeru Mukai on his sixtieth birthday, with admiration

For a $K3$ surface S, a smooth curve $C \subset S$ and a globally generated linear series $A \in W^r_d(C)$ with $h^0(C,A) = r+1$, the *Lazarsfeld-Mukai* vector bundle $E_{C,A}$ is defined via the following elementary modification on S

$$0 \longrightarrow E^\vee_{C,A} \longrightarrow H^0(C,A) \otimes \mathcal{O}_S \longrightarrow A \longrightarrow 0. \tag{1}$$

The bundles $E_{C,A}$ have been introduced more or less simultaneously in the 80's by Lazarsfeld [L1] and Mukai [M1] and have acquired quite some prominence in algebraic geometry. On one hand, they have been used to show that curves on general $K3$ surfaces verify the Brill-Noether theorem [L1], and this is still the only class of smooth curves known to be general in the sense of Brill-Noether theory in every genus. When $\rho(g,r,d) = 0$, the vector bundle $E_{C,A}$ is rigid and plays a key role in the classification of Fano varieties of coindex 3. For $g = 7, 8, 9$, the corresponding Lazarsfeld-Mukai bundle has been used to coordinatize the moduli space of curves of genus g , thus giving rise to a new and more concrete model of $\mathcal{M}_g$, see [M2], [M3], [M4]. Furthermore, Lazarsfeld-Mukai bundles of rank two have led to a characterization of the locus in $\mathcal{M}_g$ of curves lying on $K3$ surfaces in terms of existence of linear series with unexpected syzygies [F], [V]. For a recent survey on this circle of ideas, see [A].

Recently, Lazarsfeld-Mukai bundles have proven to be effective in shedding some light on an interesting conjecture of Mercat in Brill-Noether theory, see [FO1], [FO2], [LMN]. Recall that the Clifford index of a semistable vector bundle $E \in \mathcal{U}_C(n,d)$ on a smooth curve C of genus g is defined in [LN1] as

$$\gamma(E) := \mu(E) - \frac{2}{n} h^0(C,E) + 2 \geq 0.$$

Then the *higher Clifford indices* of the curve C are defined as the quantities

$$\mathrm{Cliff}_n(C) := \min\Big\{\gamma(E) : E \in \mathcal{U}_C(n,d),\ \ d \leq n(g-1),\ \ h^0(C,E) \geq 2n\Big\}.$$

For any line bundle L on C such that $h^i(C,L) \geq 2$ for $i = 0,1$, that is, contributing to the classical Clifford index $\mathrm{Cliff}(C)$, by computing the invariants of the strictly semistable vector bundle $E := L^{\otimes n}$, one finds that $\mathrm{Cliff}_n(C) \leq \mathrm{Cliff}(C)$. Mercat [Me1] predicted that for any smooth curve C of genus g, the following equality

$$(M_n): \ \ \mathrm{Cliff}_n(C) = \mathrm{Cliff}(C).$$

should hold. Counterexamples to (M_2) have been found on curves lying on $K3$ surfaces that are special in Noether-Lefschetz sense, see [FO1], [FO2] and [LN2]. However, (M_2) is expected to hold for a general curve of genus g, and in fact even for a curve C lying on a $K3$ surface S such that $\mathrm{Pic}(S) = \mathbb{Z}\cdot C$. For instance, it is known that statement (M_2) holds on $\mathcal{M}_{11}$ outside a certain Koszul divisor (which also admits a Noether-Lefschetz realization), see [FO2] Theorem 1.3. It has also been shown that (M_2) holds generically on $\mathcal{M}_g$ for $g \leq 16$, see [FO1].

It has been proved in [LMN] that rank three restricted Lazarsfeld-Mukai bundles invalidate statement (M_3) in genus 9 and 11 respectively, that is, Mercat's conjecture in rank three fails generically on $\mathcal{M}_9$ and $\mathcal{M}_{11}$ respectively. This was then extended in [FO2] Theorem 1.4, to show that on a $K3$ surface S with $\mathrm{Pic}(S) = \mathbb{Z}\cdot C$, where $C^2 = 2g-2$, if $A \in W^2_d(C)$ is a linear system where $d := \lfloor \frac{2g+8}{3} \rfloor$, the restriction to C of the Lazarsfeld-Mukai bundle $E_{C,A}$ is stable and has Clifford index strictly less than $\lfloor \frac{g-1}{2} \rfloor$, in particular, statement (M_3) fails for the curve C. For further background on this problem, we refer to the papers [Me1], [LN1] and [GMN].

The restricted Lazarsfeld-Mukai bundle $E|_C := E_{C,A} \otimes \mathcal{O}_C$ sits in the following exact sequence on the curve $C \subset S$

$$0 \longrightarrow Q_A \longrightarrow E|_C \longrightarrow K_C \otimes A^\vee \longrightarrow 0, \tag{2}$$

where $Q_A = M_A^\vee$ is the dual of the kernel bundle defined by the sequence

$$0 \longrightarrow M_A \longrightarrow H^0(C, A) \otimes \mathcal{O}_C \longrightarrow A \longrightarrow 0.$$

One then easily shows [V], [FO2] that the sequence (2) is exact on global sections, that is,

$$h^0(C, E|_C) = h^0(C, K_C \otimes A^\vee) + h^0(C, Q_A) = g - d + 2r + 1.$$

By choosing the degree d minimal such that $W^r_d(C) \neq \emptyset$, precisely $d = r + \lfloor \frac{r(g+1)}{r+1} \rfloor$, it becomes clear that, for sufficiently high g, one has

$$\gamma(E|_C) < \mathrm{Cliff}(C),$$

that is, $E|_C$, when semistable, provides a counterexample to Mercat's conjecture (M_{r+1}). We prove the following result, extending to rank 4 a picture studied in smaller ranks in the papers [M1], [V], respectively [FO2].

Theorem 0.1. *Let S be a $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z}\cdot L$, where $L^2 = 2g-2$ and write*

$$g = 4i - 4 + \rho \ \text{ and } \ d = 3i + \rho,$$

with $\rho \geq 0$ and $i \geq 6$. Then for a general curve $C \in |L|$ and a globally generated linear series $A \in W^3_d(C)$ with $h^0(C, A) = 4$, the restriction to C of the Lazarsfeld-Mukai bundle $E_{C,A}$ is stable.

Note that in Theorem 0.1, $\dim W^3_d(C) = \rho$. The rank 3 version of this result was proved in [FO2]. We record the following consequence of Theorem 0.1:

Corollary 0.2. *For $C \subset S$ with $g \geq 20$ and $\mathrm{Pic}(S) = \mathbb{Z}\cdot C$, we set $d := \lfloor \frac{4g+14}{3} \rfloor$ and $A \in W^3_d(C)$ with $h^0(C, A) = 4$. Then $E|_C$ is a stable rank 4 bundle with $\gamma(E|_C) < \lfloor \frac{g-1}{2} \rfloor$. It follows that the statement (M_4) fails for C.*

The curves C appearing in Corollary 0.2 are Brill-Noether general, that is, they satisfy $\mathrm{Cliff}(C) = \lfloor \frac{g-1}{2} \rfloor$, see [L1].

Theorem 0.1 and Corollary 0.2 fit into a more general set of results that are independent from the structure of $\mathrm{Pic}(S)$. For example, we show that under mild restrictions, on a very general $K3$ surface, the extension (2) is non-trivial and the restricted Lazarsfeld-Mukai bundle $E|_C$ is simple (see Theorem 1.3). We expect that the bundle $E|_C$ remains stable also for higher rank $r+1 = h^0(C, A)$, at least when $\mathrm{Pic}(S) = \mathbb{Z}\cdot C$. However, our method of proof based on the Bogomolov inequality, seems not to extend easily for $r \geq 4$.

The second topic we discuss in this paper concerns a connection between normal bundles of canonical curves and Mercat's conjecture. The question we pose is however fundamental and interesting irrespective of Mercat's conjecture.

For a smooth non-hyperelliptic canonically embedded curve $C \subset \mathbf{P}^{g-1}$ of genus g, we consider the normal bundle $N_C := N_{C/\mathbf{P}^{g-1}}$; we then define the twist of the conormal bundle $E := N_C^{\vee} \otimes K_C^{\otimes 2}$. By direct calculation

$$\det(E) = K_C^{\otimes(g-5)} \text{ and } \mathrm{rk}(E) = g-2.$$

In particular, the vector bundle E contributes to $\mathrm{Cliff}_{g-2}(C)$ if and only if $g \leq 8$. Since $M_{K_C}(-1) = \Omega_{\mathbf{P}^{g-1}|C}$, the bundle E sits in the following exact sequence

$$0 \longrightarrow E \longrightarrow M_{K_C} \otimes K_C \stackrel{\gamma_{K_C}}{\longrightarrow} K_C^{\otimes 3} \longrightarrow 0, \tag{3}$$

where $\gamma_{K_C} : H^0(C, M_{K_C} \otimes K_C) \to H^0(C, K_C^{\otimes 3})$ is the Gaussian map of C, see [W]. The map γ_{K_C} vanishes on symmetric tensors, hence $\mathrm{Ker}(\gamma_{K_C}) = I_2(K_C) \oplus \mathrm{Ker}(\psi_{K_C})$, where

$$\psi_{K_C} := \gamma_{K_C|_{\wedge^2 H^0(C,K_C)}} : \bigwedge^2 H^0(C, K_C) \to H^0(C, K_C^{\otimes 3}),$$

and $I_2(K_C) = K_{1,1}(C, K_C)$ is the space of quadrics containing the canonical curve C. The map ψ_{K_C} has been studied intensely in the context of deformations in $\mathbf{P}^g$ of the cone over the canonical curve $C \subset \mathbf{P}^{g-1}$, see [W]. It is in particular known [CHM], [V] that ψ_{K_C} is surjective for a general curve C of genus $g \geq 12$.

We now specialize to the case $g = 7$, when E contributes to $\mathrm{Cliff}_5(C)$. Then

$$\mathrm{rk}(E) = 5 \text{ and } \det(E) = K_C^{\otimes 2},$$

therefore $\mu(E) = \frac{24}{5}$. It is easy to show that the Gaussian map ψ_{K_C} is injective for every smooth curve C of genus 7 having maximal Clifford index $\mathrm{Cliff}(C) = 3$. In particular, the space

$$H^0(C, E) = I_2(K_C)$$

is 10-dimensional and $\gamma(E) = 2 + \frac{4}{5} < \mathrm{Cliff}(C)$. We establish the following result:

Theorem 0.3. *The normal bundle $N_{C/\mathbf{P}^6}$ of every canonical curve C of genus 7 with maximal Clifford index is stable. In particular, the Mercat conjecture (M_5) fails for a general curve of genus 7.*

The proof of Theorem 0.3 uses in an essential way Mukai's realisation [M3] of a canonical curve C of genus 7 with $\mathrm{Cliff}(C) = 3$ as a linear section of the 10-dimensional spinorial variety $OG(5, 10) \subset \mathbf{P}^{15}$. In particular, the vector bundle E is the restriction to C of the rank 5 spinorial bundle on $OG(5,10)$, which endows E with an extra structure that only exists in genus 7. Note that the normal bundle of every canonical curve of genus at most 6 is unstable, and more generally, the normal bundle of a tetragonal canonical curve of any genus is unstable (see also Section 3). In particular, we have the following identification of cycles on $\mathcal{M}_7$

$$\Big\{[C] \in \mathcal{M}_7 : N_C \text{ is unstable}\Big\} = \mathcal{M}^1_{7,4}, \tag{4}$$

where the right hand side denotes the divisor of tetragonal curves of genus 7. We make the following conjecture:

Conjecture 0.4. *The normal bundle $N_{C/\mathbf{P}^{g-1}}$ of a general canonical curve C of genus $g \geq 7$ is stable.*

Note that the stability of the normal bundle $N_{C/\mathbf{P}^r}$ of a curve of genus g is not known even in the case of a non-special embedding $C \hookrightarrow \mathbf{P}^r$ given by a line bundle $L \in \mathrm{Pic}(C)$ of large degree. This is in stark contrast with the case the kernel bundle $M_L = \Omega_{\mathbf{P}^r|C}(1)$, whose stability easily follows by a filtration argument due to Lazarsfeld [L2]. For some very partial results in this direction, see [EL]. In general, one can show by degenerating a canonical curve $C \subset \mathbf{P}^{g-1}$ to the transversal union of two rational normal curves in $\mathbf{P}^{g-1}$ meeting transversally in $g+1$ points, that $N_{C/\mathbf{P}^{g-1}}$ is not too unstable. Due to the fact that the slope $\mu(N_{C/\mathbf{P}^{g-1}})$ is not an integer, this simple minded technique does not seem to lead to a full solution, because one cannot expect to find a specialization in which the corresponding limit of the normal bundle is a direct sum of line bundles of the same degree. It is of course, natural to ask whether a generalization of the equality (4) exists for higher genus.

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1. SIMPLICITY OF RESTRICTED LAZARSFELD-MUKAI BUNDLES

We fix a $K3$ surface S, a smooth curve $C \subset S$ of genus g and a globally generated linear series $A \in W^r_d(C)$, with $h^0(C,A) = r+1$. Using the evaluation sequence (1), we form the vector bundle $F = F_{C,A}$; by dualizing, we obtain an exact sequence for the dual bundle $E = E_{C,A} := F^\vee_{C,A}$:

$$0 \longrightarrow H^0(C,A)^\vee \otimes \mathcal{O}_S \longrightarrow E_{C,A} \longrightarrow K_C \otimes A^\vee \longrightarrow 0. \tag{5}$$

It is well-known [M1], [L1] that $c_1(E) = [C]$ and $c_2(E) = d$; moreover $h^0(S,F) = 0$ and $h^1(S,E) = h^1(S,F) = 0$. Finally, one also has that

$$\chi(S, E \otimes F) = 2 - 2\rho(g,r,d);$$

in particular, if E is a simple bundle, then $\rho(g, r, d) \geq 0$. Assuming furthermore that $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$, it is also well-known that both E and F are C-stable bundles on S.

1.1. **The rank 2 case.** We begin by showing that in rank 2, irrespective of the structure of $\mathrm{Pic}(S)$, a splitting of the restriction $E|_C$ can only be induced by an elliptic pencil on the $K3$ surface.

Theorem 1.1. *Let $C \subset S$ be as above and $A \in W^1_d(C)$ a base point free pencil of degree $2 < d < g-1$ with $K_C \otimes A^\vee$ globally generated. The following conditions are equivalent:*

(i) $E|_C \cong A \oplus (K_C \otimes A^\vee)$;
(ii) *There exists an elliptic pencil $N \in \mathrm{Pic}(S)$ such that $N|_C = A$.*
(iii) $h^0(S, E \otimes F) < h^0(C, E \otimes F|_C)$.

Corollary 1.2. *With notation as above, if $g \leq 2d-2$ and A is not induced by an elliptic pencil on S, then $E|_C$ is simple if and only if E is simple.*

Note that it is easy to see that if $E|_C$ is simple, then E is also simple. It is also known that if E is simple, then automatically $g \leq 2d-2$.

Proof. (of Theorem 1.1) *(ii)⇒(i).* Let N be an elliptic pencil with $N|_C = A$. Consider the exact sequence

$$0 \longrightarrow N^\vee \longrightarrow F \longrightarrow N(-C) \longrightarrow 0.$$

Its restriction to C gives a splitting of the dual of the sequence (2) characterizing $E|_C$. Observe that since $d < g-1$, there is no morphism from $A^\vee$ to $K_C^\vee \otimes A$.

(i)⇒(ii). Conversely, suppose that $E|_C = A \oplus (K_C \otimes A^\vee)$. Applying $\mathrm{Hom}(K_C \otimes A^\vee, -)$ to the sequence (1), we obtain an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(K_C \otimes A^\vee, F) \longrightarrow \mathrm{Ext}^1(K_C \otimes A^\vee, H^0(C, A) \otimes \mathcal{O}_S) \longrightarrow \mathrm{Ext}^1(K_C \otimes A^\vee, A).$$

Since the extension class $[E] \in \mathrm{Ext}^1(K_C \otimes A^\vee, H^0(C, A) \otimes \mathcal{O}_S)$ maps to the trivial extension in $\mathrm{Ext}^1(K_C \otimes A^\vee, A)$, it follows that there exists a rank 2 bundle G on S which fits into a commutative diagram:

(6)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & H^0(A) \otimes \mathcal{O}_S & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_C \otimes A^\vee & = & K_C \otimes A^\vee & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Using that $H^0(S, F) = H^1(S, F) = 0$, we obtain $H^0(S, G) \cong H^0(C, K_C \otimes A^\vee)$. Since $h^0(S, E) = h^0(C, A) + h^1(C, A) = h^0(C, A) + h^0(S, G)$, and $h^1(S, E) = 0$, it follows that $H^1(S, G) = 0$. From the second row of (6), we find that $H^0(S, G(-C)) = 0$.

Furthermore, we compute $c_1(G) = 0$ and $c_2(G) = 2d - 2g + 2$. So $c_2(G) < 0 = c_1^2(G)$, that is, G violates Bogomolov's inequality, and then it sits in an extension

$$0 \longrightarrow M \longrightarrow G \longrightarrow M^\vee \otimes \mathcal{I}_{\Gamma/S} \longrightarrow 0, \tag{7}$$

where Γ is a zero-dimensional subscheme of S, and $M \in \mathrm{Pic}(S)$ is such that $M^2 > 0$ and $M \cdot H > 0$ for any ample line bundle H on S. In particular, $H^0(S, M^\vee) = 0$, and hence $H^0(S, M) \cong H^0(S, G) \cong H^0(C, K_C \otimes A^\vee) \neq 0$. Moreover, since

$$h^0(S, M^\vee \otimes \mathcal{I}_{\Gamma/S}) = h^1(S, G) = 0,$$

it also follows that $H^1(S, M) = 0$.

On the other hand $H^0(S, F) = 0$, which implies that the composed map

$$M \longrightarrow G \longrightarrow K_C \otimes A^\vee$$

is non-zero; in fact, we claim that it is surjective, that is, $M|_C = K_C \otimes A^\vee$. Suppose that $M|_C = K_C \otimes A^\vee(-D')$, with $D' \neq 0$ an effective divisor on C. Since $h^0(S, G(-C)) = 0$, we have $h^0(S, M(-C)) = 0$, which implies $h^0(S, M) \leq h^0(C, M|_C)$. Since we assumed $K_C \otimes A^\vee$ to be globally generated, we have that

$$h^0(S, M) \leq h^0(C, K_C \otimes A^\vee(-D')) < h^0(C, K_C \otimes A^\vee) = h^0(S, M),$$

a contradiction.

Setting $N := M^\vee(C)$, we have shown that $N|_C = A$ and there is an exact sequence

$$0 \longrightarrow M^\vee \longrightarrow N \longrightarrow A \longrightarrow 0.$$

Since $h^0(S, M^\vee) = h^1(S, M^\vee) = 0$, it follows that $h^0(S, N) = h^0(C, A) = 2$ and hence N defines an elliptic pencil.

(iii)⇒(i). From the sequence (1) twisted by $E(-C) \cong F$, we obtain that

$$H^0(S, E \otimes F(-C)) \subset H^0(C, A) \otimes H^0(S, E(-C)),$$

and, since F has no sections, it follows that $H^0(S, E \otimes F(-C)) = 0$. We have an exact sequence

$$0 \longrightarrow H^0(S, E \otimes F) \longrightarrow H^0(S, E \otimes F|_C) \longrightarrow H^1(S, E \otimes F(-C)).$$

The hypothesis implies that $H^1(S, E \otimes F(-C)) \neq 0$. From (1) twisted by $E(-C) \cong F$, we obtain the exact sequence in cohomology

$$0 \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^1(S, E \otimes F(-C)) \longrightarrow H^0(C, A) \otimes H^1(S, E(-C)) = 0,$$

therefore $h^0(C, E|_C \otimes K_C^\vee \otimes A) \neq 0$. The sequence (2) yields to an exact sequence

$$0 = H^0(C, K_C^\vee \otimes A^{\otimes 2}) \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^0(C, \mathcal{O}_C) \to H^1(C, K_C^\vee \otimes A^{\otimes 2}).$$

Then $H^0(C, E|_C \otimes K_C^\vee \otimes A) \to H^0(C, \mathcal{O}_C)$ is an isomorphism and, under the coboundary map

$$H^0(C, \mathcal{O}_C) \ni 1 \mapsto 0 \in H^1(C, K_C^\vee \otimes A^{\otimes 2}),$$

that is, the sequence (2) is split.

Note that we also have $h^1(S, E \otimes F(-C)) = 1$ and $h^0(C, E \otimes F|_C) = h^0(S, E \otimes F) + 1$.

(i)⇒(iii). From the hypothesis and from the sequence (2), we find

$$h^0(C, E|_C \otimes A^\vee) = h^0(C, K_C \otimes A^{\otimes(-2)}) + 1.$$

Furthermore, $h^0(S, E \otimes F) = h^0(C, E|_C \otimes A^\vee)$; twist (5) by F and use the vanishing of $h^0(F)$ and that of $h^1(F)$.

On the other hand, since $E|_C \cong A \oplus K_C \otimes A^\vee$, we have

$$h^0(C, E \otimes F|_C) = 2 + h^0(C, K_C \otimes A^{\otimes(-2)}),$$

hence $h^0(C, E \otimes F|_C) = h^0(S, E \otimes F) + 1$. □

1.2. **Lazarsfeld-Mukai bundles of higher rank.** We study when the restriction $E|_C$ is a simple vector bundle. Our main tool is a variant of the Bogomolov instability theorem.

Theorem 1.3. *Let S be a $K3$ surface and $C \subset S$ a smooth curve of genus $g \geq 4$ such that $\mathrm{Pic}(S) = \mathbb{Z} \cdot C$. We fix positive integers r and d such that*

$$\rho(g, r, d) \geq 0,\ g \geq 2r + 4 \text{ and } d \leq \frac{3r(g-1)}{2r+2}.$$

Then for any linear series $A \in W^r_d(C)$ such that $h^0(C, A) = r + 1$ and $K_C \otimes A^\vee$ is globally generated, the restricted Lazarsfeld-Mukai bundle $E|_C$ is simple.

Note that in the special case $\rho(g, r, d) = 0$, the constraints from the previous statement give rise to the bound $g > 2r + 5$.

Proof. Step 1. We first establish that the natural extension (2), that is,

$$0 \longrightarrow Q_A \longrightarrow E|_C \longrightarrow K_C \otimes A^\vee \longrightarrow 0$$

is non-trivial. Assuming that (2) is trivial. Then there is an injective morphism from $K_C \otimes A^\vee$ to $E|_C$ and hence a surjective map $F(C) \to A$. Then

$$G := \mathrm{Ker}\{F(C) \to A\}$$

is a vector bundle of rank $r + 1$ with Chern classes $c_1(G) = (r - 1)[C]$ and

$$c_2(G) = c_2(F(C)) - c_1(F(C)) \cdot C + \deg(A) = 2d + r(r - 3)(g - 1).$$

We compute the discriminant of G

$$\Delta(G) = 2\mathrm{rk}(G)c_2(G) - (\mathrm{rk}(G) - 1)c_1^2(G) = 4d(r + 1) - 8r(g - 1) < 0,$$

hence G is unstable. Applying [HL] Theorem 7.3.4, there exists a subsheaf $M \subset G$ with

$$\xi^2_{M,G} \geq -\frac{\Delta(G)}{r(r+1)^2},$$

where $\xi_{M,G} = c_1(M)/\mathrm{rk}(M) - c_1(G)/\mathrm{rk}(G)$. Setting $c_1(M) = k \cdot [C]$ and $s := \mathrm{rk}(M)$, the previous inequality becomes

$$\left(\frac{k}{s} - \frac{r-1}{r+1}\right)^2 (2g - 2) \geq \frac{8r(g-1) - 4d(r+1)}{r(r+1)^2}.$$

Note that M destabilizes G, which coupled with the stability of $F(C)$ yields

$$\frac{r-1}{r+1} \leq \frac{k}{s} < \frac{r}{r+1},$$

implying after manipulations $2d(r + 1) > 3(g - 1)r$, thus contradicting the hypothesis.

Step 2. Assuming that $E|_C$ is non-simple, we deduce that the extension (2) splits. We consider the exact sequence

$$H^0(S, E \otimes F) \longrightarrow H^0(C, E \otimes F|_C) \longrightarrow H^1(S, E \otimes F(-C)).$$

and it suffices to show that $H^1(S, E \otimes F(-C)) = 0$. Assuming this not to be the case, twisting (1) by $E(-C)$ induces the exact sequence

$$H^0(C, A \otimes E|_C \otimes K_C^\vee) \longrightarrow H^1(S, E \otimes F(-C)) \longrightarrow H^0(C, A) \otimes H^1(S, E(-C)).$$

Since $H^1(S, E(-C)) = 0$, we obtain that $H^0(C, A \otimes E|_C \otimes K_C^\vee) \neq 0$. Furthermore, Q_A is a stable bundle and since $\mu(Q_A \otimes A \otimes K_C^\vee) < 0$, we find that

$$H^0(C, Q_A \otimes A \otimes K_C^\vee) = 0,$$

hence we also have the sequence induced from (2) after twisting with $A \otimes K_C^\vee$

$$0 \longrightarrow H^0(C, E|_C \otimes K_C^\vee \otimes A) \longrightarrow H^0(C, \mathcal{O}_C) \longrightarrow H^1(C, K_C^\vee \otimes A \otimes Q_A).$$

We conclude that the coboundary map $H^0(C, \mathcal{O}_C) \to H^1(C, Q_A \otimes A \otimes K_C^\vee)$ is trivial, that is, $E|_C \cong Q_A \oplus (K_C \otimes A^\vee)$, which completes the proof. □

2. Stability of restricted Lazarsfeld-Mukai bundles

2.1. **The rank 2 case.** If $C \subset S$ is an ample curve, then with one exception ($g = 10$ and C a smooth plane sextic), Cliff(C) is computed by a pencil, see [CP] Proposition 3.3. We show that in rank 2 the semistability of the LM bundle is preserved under restriction.

Theorem 2.1. *Let S be a $K3$ surface, $C \subset S$ an ample curve of genus $g \geq 4$ and $A \in W^1_d(C)$ a pencil computing* Cliff(C). *If $E_{C,A}$ is C-semistable on S, then $E|_C$ is also semistable on C. Moreover, if $E_{C,A}$ is C-stable on S, then $E|_C$ is stable on C.*

Proof. The proof of the stability is similar, and hence we discuss the semistability part only. We write $A = \mathcal{O}_C(D)$, where D is an effective divisor on C. Suppose $E|_C$ is unstable and consider an exact sequence

$$0 \longrightarrow L_1 \longrightarrow E|_C \longrightarrow K_C \otimes L_1^\vee \longrightarrow 0,$$

with $\deg(L_1) \geq g$. Since $L_1 \nsubseteq A$, the composed map $L_1 \to E|_C \to K_C \otimes A^\vee$ must be non-zero, that is, $L_1 = K_C(-D - D_1)$, where D_1 is an effective divisor on C. Set $d_1 := \deg(D_1)$. Consider the elementary modification

$$0 \longrightarrow V \longrightarrow E \longrightarrow A(D_1) \longrightarrow 0 \tag{8}$$

induced by the composition $E \to E|_C \to A(D_1)$. Then

$$c_1(V) = 0 \text{ and } c_2(V) = 2d + d_1 - 2g + 2 < 0,$$

hence V is unstable with respect to any polarization and fits in an exact sequence

$$0 \longrightarrow M \longrightarrow V \longrightarrow M^\vee \otimes \mathcal{I}_{\Gamma/S} \longrightarrow 0, \tag{9}$$

where $\Gamma \subset S$ is a 0-dimensional subscheme and M is a divisor class that intersects positively any ample class on S and with $M^2 > 0$. From (8) and (9) we find that $H^0(S, M) \cong H^0(S, V)$ and $H^0(S, M(-C)) = 0$. Dualizing (8), we obtain the sequence

$$0 \longrightarrow F \longrightarrow V^\vee \longrightarrow K_C(-D - D_1) \longrightarrow 0,$$

from which, using that $V \cong V^\vee$, we obtain $H^0(S, V) = H^0(C, K_C(-D - D_1))$.

We claim that $\mathrm{Cliff}(A(D_1)) = \mathrm{Cliff}(C)$. Recall that $h^0(S,E) = h^0(C,A) + h^1(C,A)$, and, from the sequence (8) we write

$$h^0(S,E) \le h^0(C,A(D_1)) + h^1(C,A(D_1)).$$

By assumption, the pencil A computes $\mathrm{Cliff}(C)$, which implies

$$\mathrm{Cliff}(C) = g+1-h^0(A)-h^1(A) \ge g+1-h^0(A(D_1))-h^1(A(D_1)) = \mathrm{Cliff}(A(D_1)).$$

It follows that $\mathrm{Cliff}(A(D_1)) = \mathrm{Cliff}(C)$, in particular $K_C(-D-D_1)$ is globally generated.

Clearly, $M \nsubseteq F$, hence the composition $\varphi : M \to V \to K_C(-D-D_1)$ is non-zero and one writes $\mathrm{Im}(\varphi) = K_C(-D-D_1-D_2)$, where D_2 is an effective divisor on C. If $D_2 \ne 0$, then one has the sequence of inequalities

$$h^0(S,M) \le h^0(C,K_C(-D-D_1-D_2)) < h^0(C,K_C(-D-D_1)) = h^0(S,M),$$

a contradiction. Therefore $M|_C = K_C(-D-D_1)$. Viewing M as a subsheaf of E, we find $\mu(M) = M\cdot C = \deg(L_1) > \mu(E)$, thus bringing the proof to an end. □

Remark 2.2. If $E_{C,A}$ is stable, then it is simple and hence $d = \lfloor\frac{g+3}{2}\rfloor$, see [L1]. Conversely, if $C' \subset S$ is an ample curve of genus g and gonality $\lfloor\frac{g+3}{2}\rfloor$, then it was shown in [LC] that the LM bundle $E_{C,A}$ corresponding to a general curve $C \in |\mathcal{O}_S(C')|$ and a pencil $A \in W^1_{\lfloor\frac{g+3}{2}\rfloor}(C)$ is C-semistable (even stable when g is odd).

2.2. **Stability of Lazarsfeld-Mukai bundles of rank four.** We show that restrictions of LM bundles of rank 4 on very general $K3$ surfaces of genus $g \ge 20$ are stable. Similar results were established in [V] and [FO2] for rank 2 and 3 respectively. We fix integers $i \ge 6$ and $\rho \ge 0$ and write

$$g := 4i-4+\rho \quad \text{and} \quad d := 3i+\rho,$$

so that $\rho(g,3,d) = \rho$. Let S be a $K3$ surface and $C \subset S$ a curve of genus g such that $\mathrm{Pic}(S) = \mathbb{Z}\cdot C$, and pick a globally generated linear series $A \in W^3_d(C)$ with $h^0(C,A) = 4$.

Proof of Theorem 0.1. Our previous results show that $E|_C$ is simple, hence indecomposable. Suppose $E|_C$ is not stable and fix a maximal destabilizing sequence

$$0 \longrightarrow M \longrightarrow E|_C \longrightarrow N \longrightarrow 0.$$

Put $d_N := \deg(N)$ and $d_M := \deg(M) = 2g-2-d_N$. Since M is destabilizing,

$$\frac{d_M}{\mathrm{rk}(M)} \ge \frac{g-1}{2}, \quad \frac{d_N}{\mathrm{rk}(N)} \le \frac{g-1}{2}. \tag{10}$$

The bundle N, being a quotient of E, is globally generated. Since $H^0(C,E|_C^\vee) = 0$, clearly $N \ne \mathcal{O}_C$, therefore $h^0(C,N) \ge 2$. From the inequalities (10) it follows that $\mathrm{rk}(N) > 1$, because C has maximal gonality.

Step 1. We prove that M is a line bundle. Assume that, on the contrary,

$$\mathrm{rk}(M) = \mathrm{rk}(N) = 2$$

and consider the elementary modification $G := \mathrm{Ker}\{E \to N\}$. Its Chern classes are given as follows:

$$c_1(G) = -[C], \quad c_2(G) = d + d_N - 2(g-1),$$

and its discriminant equals $\Delta(G) = -64i + 110 + 8d_N - 14\rho < 0$, because of (10). In particular, there exists a saturated subsheaf $F \subset G$ which verifies the inequalities

$$\mu(G) \leq \mu(F) < \mu(E), \quad \text{and} \tag{11}$$

$$\xi^2_{F,G} \geq -\frac{\Delta(G)}{48}. \tag{12}$$

Write $c_1(F) = \alpha \cdot [C]$ and $\mathrm{rk}(F) = \beta \leq 3$. The above inequality (12) becomes

$$\left(\frac{\alpha}{\beta} + \frac{1}{4}\right)^2 (2g-2) \geq -\frac{\Delta(G)}{48}.$$

We apply (11) for $\mu(F) = \alpha(2g-2)/\beta$ and obtain

$$-\frac{1}{4} \leq \frac{\alpha}{\beta} < \frac{1}{4},$$

hence $\alpha = 0$, and the inequality (12) reads in this case $d_N \geq 5i - 10 + \rho$. Recalling that $d_N \leq g - 1 = 4i - 5 + \rho$, we obtain a contradiction whenever $i \geq 6$.

Step 2. We construct an elementary modification, in order to reach a contradiction.

From (10), we have $d_M \geq \frac{g-1}{2}$. The composite map $M \to E|_C \to K_C \otimes A^\vee$ is not zero, for else $M \hookrightarrow Q_A$ and since $\mu(Q_A \otimes M^\vee) < 0$, one contradicts the semistability of Q_A. We set $A_1 := K_C \otimes A^\vee \otimes M^\vee$ and obtain a surjection $F(C)|_C \to A \otimes A_1$ inducing, as before, an elementary modification

$$V := \mathrm{Ker}\{F(C) \to A \otimes A_1\}.$$

By direct computation we show that $\Delta(V) < 0$. Indeed, we compute

$$c_1(V) = 2 \cdot [C], \quad c_2(V) = d + 2g - 2 - d_M, \quad \text{hence}$$

$$\Delta(V) = 8c_2(V) - 3c_1^2(V) = 8(d - d_M - g + 1) = 8(5 - d_M - i) < 0.$$

We obtain a destabilizing sheaf $P \subset V$, with $\mathrm{rk}(P) = b \leq 3$ and $c_1(P) := a \cdot [C]$, such that the following inequalities are both satisfied

$$\left(\frac{a}{b} - \frac{1}{2}\right)^2 (2g-2) \geq -\frac{\Delta(V)}{48} \quad \text{and} \quad \mu(V) \leq \mu(P) < \mu(F(C)). \tag{13}$$

The second inequality gives $\frac{1}{2} \leq \frac{a}{b} < \frac{3}{4}$, which leaves two possibilities: either $a = 1$ and $b = 2$, when via (13) one finds that $\Delta(V) \geq 0$, a contradiction, or else $a = 2$ and $b = 3$, when inequalities (13) and (10) clash. □

3. Normal bundle of canonical curves of genus 7

The aim of this section is to prove Theorem 0.3 and we begin by recalling Mukai's results [M3] on canonical curves of genus 7. We choose a vector space $U := \mathbb{C}^{10}$ and a non-degenerate quadratic form $q : U \to \mathbb{C}$, defining a smooth 8-dimensional quadric $Q \subset \mathbf{P}(U) = \mathbf{P}^9$.

The algebraic group $\mathbf{Spin}(U)$ corresponding to the Dynkin diagram D_5 admits two 16-dimensional half-spin representations $\mathcal{S}^+$ and $\mathcal{S}^-$, which correspond to maximal weights α^+ and α^- respectively. The homogeneous spaces $V^\pm := \mathbf{Spin}(U)/P(\alpha^\pm)$ are both 10-dimensional and can be realized as the two irreducible components of the

Grassmannian $G_q(5, U)$ of projective 4-planes inside $\mathbf{P}(U)$ which are isotropic with respect to the quadratic form q. From now on, we set

$$V := V^+ \subset \mathbf{P}(\mathcal{S}^+) = \mathbf{P}^{15}.$$

Note that $\mathrm{Aut}(V) = SO(10)$. If $\mathcal{E}$ is the restriction to V of the tautological bundle on $G(5, 10)$, one has an exact sequence of vector bundles on V:

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow U \otimes \mathcal{O}_V \longrightarrow \mathcal{E} \longrightarrow 0. \tag{14}$$

By the adjunction formula, smooth curvilinear sections of V are canonical curves of genus 7 and Mukai [M3] showed that *each* curve $[C] \in \mathcal{M}_7$ with $\mathrm{Cliff}(C) = 3$ appears in this way. Precisely, there is a birational map

$$\alpha : G(7, 16)//SO(10) \dashrightarrow \overline{\mathcal{M}}_7, \ \ \alpha(\Lambda) := [\Lambda \cap V],$$

where $\Lambda \cong \mathbf{P}^6$. Given a curve $[C] \in \mathcal{M}_7$, the inverse $\alpha^{-1}([C])$ is constructed precisely via the twist of the conormal bundle on C mentioned in the introduction.

Let $C \subset \mathbf{P}^6$ be a smooth canonical curve with $\mathrm{Cliff}(C) = 3$, and set $E := N^\vee_{C\mathbf{P}^6}(2)$. One has an identification $H^0(C, E) = I_2(K_C)$ and E is a globally generated bundle. The tautological map

$$\phi_E : C \to G(5, H^0(C, E))$$

is easily shown to be injective and its image lies on V. In particular, the vector bundle E is the restricted spinorial bundle, that is, $E = \mathcal{E}_{|C}$ and one has an exact sequence:

$$0 \longrightarrow E^\vee \longrightarrow H^0(C, E) \otimes C \longrightarrow E \longrightarrow 0. \tag{15}$$

Note that $W^1_4(C) = \emptyset$, while $W^1_5(C)$ is a curve. We are going to make essential use of the following fact:

Lemma 3.1. *Let C as above and $A \in W^1_5(C)$. Then there are no surjections $E \twoheadrightarrow A$.*

Proof. We proceed by contradiction. Assume that there is such a pencil $A \in W^1_5(C)$, then use the base point free pencil trick to write the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^\vee & \longrightarrow & H^0(C, E) \otimes \mathcal{O}_C & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^\vee & \longrightarrow & H^0(C, A) \otimes \mathcal{O}_C & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array} \tag{16}$$

In particular, $H^0(C, E \otimes A^\vee) \neq 0$. Via the identification $H^0(C, E) = I_2(K_C)$, this implies that if $L := K_C \otimes A^\vee \in W^2_7(C)$, then the multiplication map

$$\mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is not injective. This is possible only if L is not birationally very ample, in particular, C must be trigonal, which is not the case. $\square$

We are now in a position to prove that the twist E of the conormal bundle of a canonical curve of genus 7 is stable.

Proof of Theorem 0.3. Suppose that $0 \to F \to E \to M \to 0$ is a destabilizing sequence for the vector bundle E, that is, with $\mu(F) \geq \mu(E) = \frac{24}{5}$. Since E is globally generated, so is any of its quotient, in particular M too. We distinguish several possibilities, depending on the ranks that appear:

(i) rk$(F) = 4$ and M is line bundle. Then deg$(F) \geq 20$, hence deg$(M) \leq 4$. Since C is not tetragonal, $h^0(C, M) \leq 1$. Note that $M \neq \mathcal{O}_C$, for $H^0(C, E^\vee) = 0$. It follows that M is not globally generated, a contradiction.

(ii) rk$(F) = 1$ and we may assume that deg$(F) = 5$. Suppose first that $h^0(C, F) = 0$, therefore $h^0(C, K_C \otimes F^\vee) = 1$, and hence $K_C \otimes F^\vee$ is not globally generated. Since one has a surjection $E^\vee(1) \twoheadrightarrow K_C \otimes F^\vee$, we reach a contradiction by observing that $E^\vee(1)$ is globally generated. Indeed, via Serre duality, this last statement is equivalent to the equality $h^0(C, E(p)) = h^0(C, E) = 10$, for every point $p \in C$. From the exact sequence

$$0 \longrightarrow E(p) \longrightarrow M_{K_C} \otimes K_C(p) \longrightarrow K_C^{\otimes 3}(p) \longrightarrow 0,$$

we obtain that $H^0(C, E(p)) = \text{Ker}\Big\{H^0(C, M_{K_C} \otimes K_C(p)) \to H^0(C, K_C^{\otimes 3}(p))\Big\}$. The conclusion follows, since $H^0(C, M_{K_C} \otimes K_C) = H^0(C, M_{K_C} \otimes K_C(p))$.

Suppose now that $h^0(C, F) \geq 1$. The case $h^0(C, F) \geq 2$ having been discarded in the course of proving Lemma 3, we assume that $h^0(C, F) = 1$, hence $h^0(C, K_C \otimes F^\vee) = 2$. We obtain that the multiplication map

$$\text{Sym}^2 H^0(C, K_C \otimes F^\vee) \to H^0(C, K_C^{\otimes 2} \otimes F^{\otimes(-2)})$$

is not injective, which contradicts the base point free pencil trick.

(iii) rk$(F) = 3$, and then deg$(F) \geq 15$, hence deg$(M) \leq 9$. This time we may assume that F is stable. If M is not stable, we choose a line subbundle $A \subset M$ of maximal degree, which we pull-back under the surjection $E \twoheadrightarrow M$, to obtain the exact sequence

$$0 \longrightarrow G \longrightarrow E \longrightarrow M/A \longrightarrow 0.$$

We obtain that deg$(M/A) \leq$ deg$(M)/2 \leq 9/2$, that is, deg$(M/A) \leq 4$. In particular, M/A is not globally generated, which is again a contradiction, so we can assume that both F and M are stable vector bundles. Since $h^0(C, M) + h^0(C, F) \geq h^0(C, E) = 10$, the strategy is to use the fact that the Mercat statements (M_2) and (M_3) have been established for curves C of genus 7 with maximal Clifford index, that is,

$$\text{Cliff}_2(C) = \text{Cliff}_3(C) = 3,$$

see [LN3] Theorem 4.5. In particular, if both F and M contribute to their respective Clifford indices, that is, $h^0(C, F) \geq 6$ and $h^0(C, M) \geq 4$ respectively, then we write

$$\frac{9}{2} + 3 \leq \frac{3}{2}\gamma(F) + \gamma(M) = \frac{1}{2}\Big(\text{deg}(F) + \text{deg}(M)\Big) - h^0(C, F) - h^0(C, M) + 5,$$

that is, $h^0(C, F) + h^0(C, M) \leq \frac{19}{2}$, a contradiction.

Assume now that one of the bundles F or M does not contribute to its Clifford index. Since M is globally generated, $h^0(C, M) \geq 2$. We can have $h^0(C, M) = 2$, only when

$M = \mathcal{O}_C^{\oplus 2}$, which is impossible, for $\mathcal{O}_C^{\oplus 2}$ is not a direct summand of E. If $h^0(C, M) = 3$, then $\deg(M) \geq 7$, and one has equality if and only if $M = Q_L$, where $L \in W^2_7(C)$. Assuming this to be the case, we choose two points $p, q \in C$ that correspond to a node in the plane model $\phi_L : C \to \mathbf{P}^2$, that is, $A := L(-p-q) \in W^1_5(C)$. Then there is a surjection $Q_L \twoheadrightarrow A$, which by composition gives rise to a surjective morphism $E \twoheadrightarrow A$. This contradicts Lemma 3.

Thus we may assume that $\deg(M) \geq 8$, and accordingly, $\deg(F) \leq 16$. Then we compute

$$\gamma(F) = \mu(F) - \frac{2}{3}h^0(C, F) + 2 \leq \frac{16}{3} - \frac{14}{3} + 2 < \mathrm{Cliff}(C),$$

which again contradicts the equality $\mathrm{Cliff}_3(C) = 3$.

(iv) $\mathrm{rk}(F) = 2$, and then $\deg(F) \geq 10$ and $\deg(M) \leq 14$. We may assume this time that M is stable. If F is not stable, then it has a line subbundle $A \hookrightarrow F$ with $\deg(A) \geq 5$, and we are back to case (ii). Thus both M and F are stable bundles, and we proceed precisely like in case (iii).

□

It is instructive to remark that the normal bundle of a canonical curve of genus $g < 7$ is never stable. More generally we have the following:

Proposition 3.2. *The normal bundle of a tetragonal canonical curve of genus g is unstable.*

Proof. More generally, we begin with a $k : 1$ covering $f : C \to \mathbf{P}^1$, and consider the rank $(k-1)$-vector bundle $\mathcal{F}^\vee := f_*\mathcal{O}_C/\mathcal{O}_{\mathbf{P}^1}$ on the projective line. Then $\pi : X = \mathbf{P}(\mathcal{F}) \to \mathbf{P}^1$ is a scroll of dimension $k-1$, which contains the canonical curve C and which can be embedded by the tautological bundle $\mathcal{O}_X(1)$ in $\mathbf{P}^{g-1}$ as a variety of degree $g-k+1$. Denoting by $H, R \in \mathrm{Pic}(X)$ the class of the hyperplane section and that of the ruling respectively, we have

$$K_X \equiv -(k-1)H + (g-k-1)R,$$

whereas obviously $C \cdot H = 2g-2$ and $C \cdot R = k$. We compute the degree of the normal bundle $N_{C/X}$ and find:

$$\deg(N_{C/X}) = \deg(T_{X|C}) + \deg(K_C) = k(g+k-1).$$

We write the usual exact sequence relating normal bundles

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbf{P}^{g-1}} \longrightarrow N_{X/\mathbf{P}^{g-1}} \otimes \mathcal{O}_C \longrightarrow 0,$$

and compare the slopes

$$\mu(N_{C/X}) = \frac{k(g+k-1)}{k-2} \quad \text{and} \quad \mu(N_{C/\mathbf{P}^{g-1}}) = \frac{2(g-1)(g+1)}{g-2}.$$

We conclude that for $k = 4$ and $g \geq 6$, the normal bundle $N_{C/X}$ is a destabilizing subbundle of $N_{C/\mathbf{P}^{g-1}}$. For g at most 5, every canonical curve of genus g is a complete intersection which obviously produces a destabilizing line subbundle. □

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